Using Euclid to Teach Geometry

by Jason Sells, Whitefield Academy

Much has been written in the classical Christian school movement about the Great Books, the use of original sources, and the importance of Latin and Greek, but where does mathematics fit in? A study of Euclid's *Elements*, the original geometry textbook, would seem like a good place to begin. The *Elements* comprised the standard geometry text for centuries because it was recognized for the beauty and coherence of its logical structure. In this article I will discuss how excerpts from Book I of the *Elements* can be used to teach students in a standard geometry class about parallel lines.

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Book I contains 48 propositions, beginning with the construction of an equilateral triangle and ending with the Pythagorean Theorem and its converse. Including all of these while covering all of the standard content in high school geometry could be a daunting task. For this reason, I advocate starting small and selecting only a few propositions to introduce at a time. Propositions 27–34 can be taught as a self-contained unit on parallel lines, starting with alternate interior angles and ending with properties of parallelograms. This unit can be taught in conjunction with (or perhaps in place of) the material on parallel lines found in most current geometry texts.

An excellent resource for those who may be new to the *Elements* is an online translation and commentary by D.E. Joyce of Clark University at http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. Professor Joyce has provided interactive Java applets that allow the illustrations to be rotated, resized, and reflected. Professor Joyce's commentary is valuable for insight into the logical structure of the *Elements* and is especially useful for selecting additional excerpts to choose for teaching, as he describes how each proposition is used to support subsequent proofs.

First, a word about prerequisites. Ideally, students in geometry will have taken a class in logic the year before or be enrolled concurrently. If this is not the case, most current geometry texts have a brief unit on logic. At a minimum, students should know how to form conditional (if–then) statements from universal statements, how to form the converse and contrapositive of a conditional statement, and the logical relationships between them. In addition, before beginning this unit students should have learned that vertical angles are congruent, the sum of the angles with a common vertex on the same side of a line is 180 degrees, and that an exterior angle of a triangle is greater than either of its opposite interior angles. (The latter theorem is proposition 16 and can be included at the beginning of the unit if students have not studied it previously.) Students should also be able to copy angles with compass and straightedge. Finally, some useful vocabulary terms not used by Euclid include transversal, alternate interior angles, and corresponding angles. These can be defined in advance and then students can be challenged to identify where Euclid makes use of them.

An effective approach to the *Elements* is to have students read the proof of a proposition and restate it in if–then form. They should then reproduce the figures in question with compass and straightedge, and rewrite the proof using modern symbols and vocabulary. This takes quite a bit of modeling together with repetition and imitation at the beginning, but by the end of the unit students should be ready to begin tackling propositions on their own.

One principle of mathematics is that the fewer things taken as postulates the better. That is, it is best to have a limited number of *a priori* assumptions and argue from these. A modern high school geometry text may have upwards of thirty postulates, but Euclid states only five (although a number of additional facts are accepted without proof implicitly). Consequently, many of the facts that are often presented without justification in today's classes can be proved by students who are studying

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the *Elements*. Celebrating this with students can foster a well-deserved sense of accomplishment for tackling a difficult subject.

An example of such a postulate in modern texts is that if alternate interior angles of two lines cut by a transversal are congruent, then the lines are parallel. In proposition 27, Euclid gives an elegant indirect proof. This provides an opportunity to teach or review the fact that a statement and its contrapositive are logically equivalent. Starting from assuming that the lines are not parallel, students can use Euclid as a guide to prove that the alternate interior angles are not congruent.

Proposition 28 applies proposition 27 to show that if corresponding angles are congruent then lines are parallel, and that lines are parallel if the sum of interior angles on the same side is 180 degrees. Proposition 29 provides proofs of the converse of the previous theorems, and is the first to use the parallel postulate, the fifth of Euclid's postulates. This postulate states that if the sum of interior angles on the same side of a transversal is less than 180 degrees, then the lines will intersect on that side of the transversal. A discussion of the attempts to prove this postulate from the other four, and the subsequent discovery of non-Euclidean geometries, is beyond the scope of this article. However a discussion of the historical narrative of these discoveries can make for a worthwhile supplement to a first-year course. Suffice it to say that Euclid's fifth postulate may be presented at the beginning of this unit or it can be introduced before proposition 29.

Proposition 30 shows that lines that are parallel to the same line are also parallel to each other. Proposition 31 demonstrates the construction of parallel lines and proposition 32 is the triangle sum theorem. Propositions 33 and 34 concern properties of parallelograms. The proofs in this sequence include both direct and indirect reasoning, giving students opportunities to practice both. If students have sufficiently well-developed skills in rhetoric they may be able to rewrite the proofs in paragraph form, but for most classes it is more appropriate to have them write two column proofs with statements in modern symbols in one column and the supporting definitions, postulates, or theorems in the other.

One challenge of using Euclid in a geometry class is the lack of extra problems to allow students to practice new skills and concepts. However, as noted above, any current text has a well-developed unit on parallel lines. Although the order of lessons may be different, it should be possible to pick and choose problems to provide practice on the concepts in each proposition.

The Great Books are those which have stood the test of time, and Euclid's *Elements* has helped to develop students' logical thinking and spatial reasoning over the span of many centuries. I hope that this article will encourage you to investigate the *Elements* further and perhaps incorporate them into your geometry classes.